Solution to Problems ♡–8

Problem A: *Assume that a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous at an $x_0 \in \mathbb{R}$ and satisfies Cauchy functional equation:*

$$f(x+y) = f(x) + f(y) \qquad \textit{for all } x, y \in \mathbb{R}.$$

Prove that $f(x) = f(1) \cdot x$ for all $x \in \mathbb{R}$.

Answer: First note that for any $x_1, \ldots, x_n \in \mathbb{R}$ we have $f(x_1 + x_2 + \ldots + x_n) = f(x_1) + f(x_2) + \ldots + f(x_n)$ (straightforward induction on n). Hence for every $x \in \mathbb{R}$ and $n \in \mathbb{N}$ we have

$$f(nx) = f(\underbrace{x + \ldots + x}_{n \text{ times}}) = \underbrace{f(x) + \ldots + f(x)}_{n \text{ times}} = nf(x).$$

Applying the above formula to $\frac{x}{n}$ we get

$$f\Big(\frac{x}{n}\Big) = \frac{1}{n} f(x)$$

for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Consequently, if $x \in \mathbb{R}$ and $n, m \in \mathbb{N}$ then

$$f\Big(\frac{n}{m}x\Big) = nf\Big(\frac{1}{m}x\Big) = \frac{n}{m} f(x).$$

Therefore,

$(\circledast)_0$ if $r > 0$ is a rational number then $f(r) = f(r \cdot 1) = r \cdot f(1)$.

Now note that $f(0) = f(0+0) = f(0) + f(0)$, so $f(0) = 0$ and

$$0 = f(0) = f\big(x + (-x)\big) = f(x) + f(-x),$$

and hence

$(\circledast)_1$ $f(-x) = -f(x)$ for each $x \in \mathbb{R}$.

Putting this together with $(\circledast)_0$ we may conclude that

$(\circledast)_2$ $f(r) = r \cdot f(1)$ for every rational number r.

Now assume that f is continuous at $x_0 \in \mathbb{R}$ and let us argue that

$(\circledast)_3$ f is continuous at 0.

For this suppose that $\big(y_n\big)_{n=1}^{\infty}$ is a sequence converging to 0. Then the sequence $\big(x_0 + y_n\big)_{n=1}^{\infty}$ converges to x_0 and by the continuity of f at x_0 we have

$$\lim_{n\to\infty} f(x_0 + y_n) = f(x_0).$$

But

$$\lim_{n\to\infty} f(x_0 + y_n) = \lim_{n\to\infty} \Big(f(x_0) + f(y_n)\Big) = f(x_0) + \lim_{n\to\infty} f(y_n),$$

so necessarily $\lim_{n\to\infty} f(y_n) = 0$.

Now we may finalize our proof. Let $x \in \mathbb{R}$ be arbitrary. Choose a sequence $\big(r_n\big)_{n=1}^{\infty}$ of rational numbers converging to x. Then $\big(x-r_n\big)_{n=1}^{\infty}$ converges to 0, so by $(\circledast)_3$ we have

$$\lim_{n\to\infty} f(x - r_n) = f(0) = 0.$$

But,

$$f(x - r_n) = f\big(x + (-r_n)\big) = f(x) + f(-r_n) = f(x) + (-r_n) \cdot f(1),$$

so

$$0 = \lim_{n\to\infty} f(x - r_n) = \lim_{n\to\infty} \Big(f(x) - r_n \cdot f(1)\Big)$$

and $f(x) = \lim_{n\to\infty} r_n \cdot f(1) = x \cdot f(1)$.

Correct solution was received from :

(1) Cody Anderson — POW 8A: ♡
(2) Brad Tuttle — POW 8A: ♡

Problem B: *Find all continuous functions* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *satisfying the Jensen equation*

$$f\Big(\frac{x+y}{2}\Big) = \frac{f(x)+f(y)}{2} \qquad \textit{for all } x, y \in \mathbb{R}.$$

Answer: Set $f(0) = c$. Putting $y = 0$ in the Jensen equation we get that for each $x \in \mathbb{R}$,

$$f\Big(\frac{x}{2}\Big) = \frac{f(x)+f(0)}{2} = \frac{f(x)+c}{2}.$$

Hence, for all $x, y \in \mathbb{R}$

$$\frac{f(x)+f(y)}{2} = f\Big(\frac{x+y}{2}\Big) = \frac{f(x+y)+c}{2}$$

and consequently

$$f(x) + f(y) = f(x+y) + c$$

(for all $x, y \in \mathbb{R}$). Now set $g(x) = f(x) - c$ (for all $x \in \mathbb{R}$) and note that

- $g : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous, and for all x, y we have
- $g(x+y) = f(x+y) - c = f(x) + f(y) - 2c = (f(x)+c) + (f(y)+c) = g(x) + g(y)$.

Therefore, by Problem A, we may conclude that $g(x) = g(1) \cdot x$ for all $x \in \mathbb{R}$, in other words

$$f(x) = ax + c, \qquad \text{where } a = f(1) - f(0) \text{ and } c = f(0).$$

Correct solution was received from :

(1) Cody Anderson POW 8B: ♡

Problem C: *Prove that there is a function $f : \mathbb{R} \longrightarrow \mathbb{Q}$ satisfying the following three conditions:*

(1) *$f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$,*
(2) *$f(x) = x$ for all $x \in \mathbb{Q}$, and*
(3) *f is not continuous at any point on $\mathbb{R}$.*

Answer: The set (field) $\mathbb{R}$ can be regarded as a vector space over the rationals $\mathbb{Q}$. A *Hamel basis* for $\mathbb{R}$ over $\mathbb{Q}$ is a maximal linearly independent set. Choose a Hamel basis $\mathbf{H}$ that contains 1. Thus each $x \in \mathbb{R}$ can be represented in a unique way as

$$x = \sum_{h \in \mathbf{H}} w_h(x) h,$$

where only finitely many coefficients $w_h(x) \in \mathbb{Q}$ are different from zero. Consequently, for $x, y \in \mathbb{R}$,

$$x + y = \sum_{h \in \mathbf{H}} w_h(x+y) h = \sum_{h \in \mathbf{H}} \big(w_h(x) + w_h(y)\big) h,$$

which implies $w_h(x+y) = w_h(x) + w_h(y)$. So, in particular, $f = w_1$ satisfies condition (1). We will show that it has the other properties too.

Since $1 = 1 \cdot 1$, it should be clear that $w_1(1) = 1$. By the proof for Problem A (clause $(\circledast)_2$ there) we know that

- $w_1(r) = r \cdot w_1(1) = r$ for every rational number r,

as required in (2). Finally, if w_1 were continuous at some point x_0, then by Problem A we would have had $w_1(x) = x$ for every $x \in \mathbb{Q}$. However, the function w_1 assumes only rational values, so $w_1(\pi) \neq \pi$. Thus clause (3) follows.

The existence of Hamel basis requires the use of Axiom of Choice and we know it cannot be defined in any reasonable way. Intrigued? Would like to learn more about set theoretic methods in mathematics? Talk to Andrzej Rosłanowski about possible course/seminar "Set Theory for a working mathematician".

No correct solution were received